Applications in solution of physical problems by using Lagrange and Hamilton's equations for conservative systems

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Abstract

In this paper, Lagrange and Hamilton's equations for the conservative systems have been studied through the applications and solving some physical questions in Newton's field, electrical field, also of the plants moving within the sun's gravity field as well, the solving also have been compared and found all identical , and extraction of the potential energy of conservative force. But dealing with Hamilton's equations is easier because they are from the first order; therefore, we are dealing with the scalar quantities and not dealing with vector once as in Newton's mechanics.

تطبيقات حل المسائل الفيزيائية باستخدام معادلات لاغرانج وهاملتون للأنظمة المحافظة

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الخلاصة

في هذا البحث تمت دراسة معادلات لاغرانج وهاملتون للأنظمة المحافظة من خلال تطبيقات وحل بعض الأسئلة الفيزياوية في حقل نيوتن والكهربائية وكذلك في حقل حركة الكواكب في مجال جاذبية الشمس وتمت مقارنة حلول هذه الأسئلة لمعادلات لاغرانج مع معادلات هاملتون ووجدت كلها متطابقة، واستخراج الطاقة الكامنة من القوة المحافظة، ألا ان التعامل مع معادلات هاملتون يكون أسهل لأنها من الدرجة الأولى، وكذلك يكون التعامل هنا مع كميات عددية ولا نتعامل مع كميات متجهة كما هو الحال في ميكانيك نيوتن.

Introduction

The Lagrange and Hamilton equations are not different theories, but the equations are derived from Newton's laws of motion. That's where these equations enable us to resolve the issues most difficult , and that cannot be solved using Newton's laws , for example , a particle moving on the surface of a sphere , or a bead moving on the surface of a spherical or spiral , it is in order that the physicists developed two different methods to find the equations of motion which Lagrange and Hamilton's equations of conservative and constraint systems, have been studied with increasing interest because they appear in many relevant physical problems. Has been studied by Kilmister (1964,1967) [5,6] as well as by Simpson (2007) [2].

In this paper will study Lagrange equations and Hamilton's equations for to the conservative systems in the field of mechanics of Newton, electrical and movement of the planets in the Sun 's gravity field to three examples of each example contains a field of these fields above.

The Lagrange equation is a differential equation ordinary second-order, and the Hamilton equations are an extension of the Lagrange equations and provides a new method for the formulation of the equations of motion, where the Hamilton equations are ordinary differential equations of the first order and be a solution to these equations is easier than Lagrange equations. The study also has conservative systems also will pass in the next section, which from through conservative force can to find the value of the potential energy.

The theoretical side

The function of the Lagrange is a function given in terms of generalized coordinates q and generalized velocities $\dot{q}$ is given by the following equation :

$$L = T(q,\dot{q},t) - V(q,t) \qquad ....(1)$$

Where T is the kinetic energy and V the potential energy of any system can be q , $\dot{q}$ are Cartesian coordinates (x, y, z) or cylindrical coordinates (r,θ,z) or spherical coordinates (r,θ,ϕ) or any other coordinates to describe the physical system.[1]

The Lagrange's equation is given to conservative systems in the following relationship:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad ....(2)$$

The conservative systems the resultant of the forces acting on a particle or a group of particles can be derived from the potential energy function , the systems are called conservative and so are not conservative, in other words the force F is called the force conservative if:

$$\vec{F} = -\nabla V \qquad ....(3)$$

where V is a function of the potential energy in terms of the coordinates of the position , which means that the potential energy function, given the following:

$$V = V(x, y, z)$$

So force components are:

$$F_x = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y} \qquad ....(4)$$
$$F_z = -\frac{\partial V}{\partial z}$$

Thus, the generalized forces are given as follows:

$$F_i = F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i} \qquad ....(5)$$

In other words

$$F_i = -\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_i} \right) \qquad ....(6)$$

$$F_i = -\frac{\partial V}{\partial q_i} \qquad ....(7)$$

This means that in the case of conservative systems be generalized component force equal to the negative derivative of a function for the potential energy of the generalized q_k contrast to this component.[2]

The generalized momentum p_i is defined as follows:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad ....(8)$$

The Hamilton function which is given in terms of generalized coordinates q_i and generalized momentum p_i and is given by:

$$H(p,q,t) = \dot{q}_i p_i - L(q,\dot{q},t) \qquad ....(9)$$

And expresses all quantities , as well as Lagrange function of the coordinates and generalized momentum.

Here H is known the *Hamiltonian*, considered as a function of q, p and t only, the differential of H is given by:

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt \qquad ....(10)$$

But from defining equation (9) we can also write:

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt \qquad ....(11)$$

The terms in $d\dot{q}_i$ in equation (11) cancel in consequence of definition of generalized momentum and from Lagrange's equation it follows that:

$$dH = \dot{q}_i dp_i - p_i dq_i - \frac{\partial L}{\partial t}dt \qquad ....(12)$$

Comparison equation (12) with equation (10) we get:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{....(13)}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \text{....(14)}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad \text{....(15)}$$

Equation (13) and equation (14) are known as the *canonical equations of Hamilton.*

In the case of conservative systems or in the case where the potential energy does not depend on the velocity, generalized momenta can be written as follows:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial (T-V)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \qquad \text{....(16)}$$

So you write Hamilton function in the following form after the substitute equation (16) in equation (9) we get: [3]

$$H = \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - L \qquad \text{....(17)}$$

But

$$T = \frac{1}{2} m\dot{q}_i^2 \qquad \text{....(18)}$$

Taking the derivative of the equation (18) for the velocity $\dot{q}_i$ we get:

$$\frac{\partial T}{\partial \dot{q}_i} = m\dot{q}_i \qquad \text{....(19)}$$

The compensation equation (19) and equation (1) in equation (17) we get:

$$H = m\dot{q}_i^2 - T + V$$

$$H = 2T - T + V$$

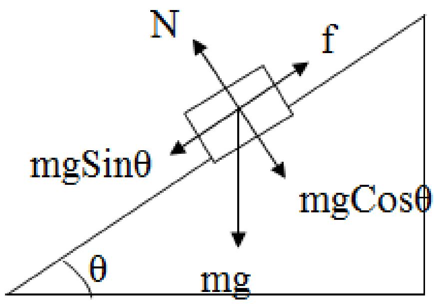
$$H = T + V = E \qquad \text{....(20)}$$

In other words, The Hamilton function is a fixed quantity represents the total energy of the conservative system, and the foregoing we conclude that if the Lagrange function does not depend on time , the function Hamilton equal quantity constant, and if the potential energy of the system does not rely on velocity, the Hamilton function equal to the total energy of the system.

Applications

1 - In the field of Newton

The first example : Body of mass m sliding down an inclined plane (with friction) and the slope angle θ on the horizon as in Figure (1), find the equation of motion for it? If the coefficient of kinetic friction between block and surface is μ?



(Figure 1)

1.1 Lagrange's Approach

The reaction force on the body on the surface is N:

$$N = mg\cos\theta \qquad ....(21)$$

Consequently , the friction force between the surface and the body are:

$$f = \mu N = \mu mg\cos\theta \qquad ....(22)$$

The resultant of component weight and the force of friction acting on the body in the dimension of x is:

$$F_x = mg\sin\theta - \mu \,\text{mg}\cos\theta \qquad ....(23)$$

Using equation (7) we get:

$$\partial V = -F_x \partial x \qquad ....(24)$$

Substitute for the value F_x from equation (23) and make the integration process on the equation (24) we get the value of the potential energy of the body:

$$V = -mgx(\sin\theta - \mu\cos\theta) \qquad ....(25)$$

The kinetic energy given by:

$$T = \frac{1}{2}m\dot{x}^2 \qquad ....(26)$$

Using equation (1) and equation (25) and equation (26) , the function of the Lagrange given by:

$$L = \frac{1}{2}m\dot{x}^2 + mgx(\sin\theta - \mu\cos\theta) \qquad ....(27)$$

Using the relationship (2) we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \qquad ....(28)$$

Extract from each , $\left(\frac{\partial L}{\partial \dot{x}}\right)$, $\left(\frac{\partial L}{\partial x}\right)$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad ....(29)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad ....(30)$$

$$\frac{\partial L}{\partial x} = mg\,(\sin\theta - \mu\cos\theta) \qquad ....(31)$$

Substitute equation (30) and equation (31) in equation (28) we get:

$$\ddot{x} = g\,(\sin\theta - \mu\cos\theta) = \mathrm{a} \qquad ....(32)$$

Where a is the value of the acceleration and the equation (32) represents the equation of motion for the body if the surface with friction.

2.1. Hamilton's Approach

It is the definition of the generalized momentum from equation (8) we get:

$$p = \frac{\partial L}{\partial \dot{x}} = \mathrm{m}\dot{x}$$

And from it we can write:

$$\dot{x} = \frac{p}{m} \qquad ....(33)$$

From equation (9) we can get the Hamilton function following :

$$H = \dot{x}p - \frac{1}{2}m\dot{x}^2 - mgx\,(\sin\theta - \mu\cos\theta) \quad ....(34)$$

Substitute equation (33) in equation (34) we get:

$$H = \frac{p^2}{2m} - mgx(\sin\theta - \mu\cos\theta) \qquad ....(35)$$

Using the equations of motion the canonical equation (13) and equation (14) we can write the equations of motion as follows:

$$\dot{x} = \frac{p}{m} \qquad ....(36)$$

$$\dot{p} = mg(\sin\theta - \mu\cos\theta) \qquad ....(37)$$

Derivative the equation (36) respect to time and then make up the value $\dot{p}$ from the equation (37) we get :

$$\ddot{x} = g(\sin\theta - \mu\cos\theta) = a \qquad ....(38)$$

Where a represents the value of the acceleration and the equation (38) is the equation of motion for the body if the surface with friction it is equivalent to the equation (32) which is the same result that we get the mechanics of Newton and obtained in an approach Lagrange.

2 - In the field of electrical

The second example : an electron moving in an electric field regularly? find acceleration to this electron ?

2.1. Lagrange's Approach

Given conservative electrical force to this electron:

$$F_{elec} = eE \qquad ....(39)$$

Where E is the intensity of the electric field and e is the electron charge. Using equation (7) we can get:

$$\partial V = -F_{elec}\partial x \qquad ....(40)$$

Substitute for the value F_{elec} from equation (39) and make the integration process on the equation (40) we get the value of the potential energy:

$$V = -eEx \qquad ....(41)$$

Its kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 \qquad ....(42)$$

Using equation (1) and equation (41) and equation (42) , the Lagrange function given by:

$$L = \frac{1}{2}m\dot{x}^2 + eEx \qquad ....(43)$$

Using the relationship (2) we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \qquad ....(44)$$

Extract from each , $\left(\frac{\partial L}{\partial \dot{x}}\right)$, $\left(\frac{\partial L}{\partial x}\right)$

$$\frac{\partial L}{\partial x} = eE \qquad ....(45)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad ....(46)$$

Derivative the equation (46) respect to time we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad ....(47)$$

Substitute equation (47) and equation (45) in equation (44) we get:

$$m\ddot{x} - eE = 0 \qquad ....(48)$$

$$\ddot{x} = \frac{eE}{m} = a \qquad ....(49)$$

The equation (49) is the value of the electron acceleration .

2.2. Hamilton's Approach

The function of the Lagrange given by equation (43)

$$L = \frac{1}{2}m\dot{x}^2 + eEx \qquad ....(50)$$

It is the definition of the generalized momentum from equation (8) we get

$$p = \frac{\partial L}{\partial \dot{x}} = \mathrm{m}\dot{x}$$

And from it we can write:

$$\dot{x} = \frac{p}{m} \qquad ....(51)$$

From equation (9) we can get the Hamilton function following:

$$H = p\dot{x} - \frac{1}{2}m\dot{x} - eEx \qquad ....(52)$$

Substitute equation (51) in equation (52) we get:

$$H = \frac{p^2}{2m} - eEx \qquad ....(53)$$

Using the equations of motion the canonical equation (13) and equation (14) we can write the equations of motion as follows:

$$\dot{x} = \frac{p}{m} \qquad ....(54)$$

$$\dot{p} = eE \qquad ....(55)$$

Derivative the equation (54) respect to time and then make up the value $\dot{p}$ from the equation (55) we get:

$$\ddot{x} = \frac{eE}{m} = a \qquad ....(56)$$

The equation (56) is equivalent to the equation (49) in the Lagrange approach above.

3 - In the field of motion of the planets in the Sun's gravity field

The three example : planet mass m moving on elliptical orbit in the Sun's gravity field strongly cohesion force gravity, find the equations of motion? Using polar coordinates.

3.1. Lagrange's Approach

Given the kinetic energy of the Cartesian coordinates the following equation:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad ....(57)$$

And the laws of conversion from Cartesian coordinates to polar coordinates are:

$$x = r\cos\theta \qquad ....(58)$$

$$y = r\sin\theta \qquad ....(59)$$

Derivative the equation (58) and equation (59) and substitute in equation (57) we get the value of the kinetic energy of the polar coordinates:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad ....(60)$$

The force of gravity is:

$$F = -\frac{GMm}{r^2} \qquad ....(61)$$

Where M is the mass of the sun, and G is the gravitational constant, using equation (7) and equation (61) we get:

The value of the potential energy:

$$V = -\frac{GMm}{r} \qquad ....(62)$$

Using equation (1) , the Lagrange function given by:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r} \qquad ....(63)$$

In this case, the generalized coordinates are r and θ so there two equations of motion, the first equation used the relationship (2) we get:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) - (\frac{\partial L}{\partial r}) = 0 \qquad .....(64)$$

Extract all of the , $(\frac{\partial L}{\partial r})$ ، $(\frac{\partial L}{\partial \dot{r}})$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{GMm}{r^2} \qquad ....(65)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad ....(66)$$

Derivative the equation (66) with respect to time we get:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) = m\ddot{r} \qquad ....(67)$$

Substitute equation (65) and equation (67) in equation (64) we get:

$$\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r} = 0 \qquad ....(68)$$

The equation (68) is the equation of motion of the planet.
The second Lagrange equation is:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - (\frac{\partial L}{\partial \theta}) = 0 \qquad ....(69)$$

Extract all of the , $(\frac{\partial L}{\partial \theta})$ ، $(\frac{\partial L}{\partial \dot{\theta}})$

$$\frac{\partial L}{\partial \theta} = 0 \qquad .....(70)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad ....(71)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = mr^2\ddot{\theta} \qquad ....(72)$$

substitute equation (72) and equation (70) in equation (69) we get:

$$mr^2\ddot{\theta} = 0 \qquad ....(73)$$

Make the integration process on the equation (73) we get:

$$mr^2\dot{\theta} = constant \qquad .....(74)$$

4.2. Hamilton's Approach

The function of the Lagrange given by equation (63):

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r} \qquad ....(75)$$

The generalized coordinates are θ and r the generalized coordinates associated with these coordinates, respectively, which can be found from the definition of linear momentum of the equation (9) we get:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad ....(76)$$

And from it we can write:

$$\dot{r} = \frac{p_r}{m} \qquad ....(77)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad ....(78)$$

And from it we can write:

$$\dot{\theta} = \frac{p_\theta}{mr^2} \qquad ....(79)$$

From equation (9) we can get the Hamilton function following:

$$H = \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} \qquad ....(80)$$

Substitute for the value of $\dot{r}$ from equation (77) and value of $\dot{\theta}$ from equation (79) in equation (80) we get:

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{GMm}{r} \qquad ....(81)$$

Using the equations of motion the canonical equation (13) and equation (14) we can write the equations of motion as follows:

$$\dot{r} = \frac{p_r}{m} \qquad ....(82)$$

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{GMm}{r^2} \qquad ....(83)$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \qquad ....(84)$$

Derivative the equation (82) with respect to time and substitute $\dot{p}_r$ from equation (83) we get:

$$\ddot{r} = \frac{p_\theta^2 - GMm^2r}{m^2r^3} \qquad ....(85)$$

and substitute p_θ from equation (78) in equation (85) we get:

$$\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r^2} = 0 \qquad ....(86)$$

The equation (86) equivalent to equation (68) in the Lagrange approach.

Results and Discussion

After we reviewed the two approaches in solving issues physical questions is the Lagrange and Hamilton for the three examples were not studied before in these two methods of three fields in physics, is the field of Newton, electric field and of planets moving within the sun's gravity field as well, and the results were all correspondent between the two methods and compare the solutions with what is in the mechanics books, we determined that the equations of Hamilton is the easiest approach from Lagrange approach in solving these examples because it is a differential equation of the first order, and the Lagrange approach it is a differential equation of the second order, as well as we were able to extraction of the potential energy of conservative force, according to equation (7) . And also concluded there was no need to deal with the vector quantities but we have to deal with scalar quantities , which removes the complexity while accessing to the solution.

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