Approximate Solution of Singular Integral Equation by Sinc Function

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Abstract:

In this paper we present a method for numerical solution of singular integral equation with Cauchy kernel of first kind using sinc-collocation method. Sinc functions are used to drive an approximate solution. Moreover, an absolute error is computed for the numerical solution.

1-Introduction :

First kind Singular integral equation with Cauchy kernel over a finite interval can be represented by:

$$\int_{-1}^{1} \frac{k(x,t)\varphi(t)}{t-x}\,dx = f(x) \quad -1 < x < 1 \qquad (1)$$

Where k(x,t) and f(x) are known continuous functions .The singular In equation (1) is interpreted as Cauchy principle value. Many application of equation (1) and other different forms have been studid in different places[1 , 7]. The theory of equation (1) is well known and it is presented in [4 , 9].Numerical solutions of equation (1)has presented in[6 , 10].

2- Sinc functions

The sinc function is defined on the whole real line by [3],

$$sinc(x) = f(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad (2)$$

Now, for h>0 and integer k, we define k[th] sinc function with step size by

$$s(k,h)(x) = \frac{\sin(\frac{\pi(x-\pi h)}{h})}{(\frac{\pi(x-\pi h)}{h})} \qquad (3)$$

On a finite interval [a,b] the basis functions are given by:

$$s(k,h)^{\circ}\mu(x) = \frac{\sin(\frac{\pi(\mu(x)-\pi h)}{h})}{(\frac{\pi(\mu(x)-\pi h)}{h})} \qquad (4)$$

Now at points x_k=kh sinc function for interpolation becomes :

$$s(k,h) = \delta_{kj}^{0} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \qquad (5)$$

So, $s(k,h)^{\circ}\mu(x)$ shows kronecker delta behavior at the grid points [5],

$$x_h = \mu(x) = \frac{a + be^{kh}}{1 + e^{kh}} \qquad (6)$$

And also approximation with sinc function for $\varphi(x)$ over interval [a,b] is

$$\varphi(x) \approx \sum_{k \pm N}^{N} \varphi(x) s(k,h)^{\circ}\mu(x) \qquad (7)$$

Where

$$\int_a^b \mu(x)\, dx \approx h \sum_{k=-N}^{N} \frac{\varphi(x_h)}{\mu'(x)} \qquad (8)$$

3- Approximate solution:

For solving equation (1) with sinc approximation we need to choose a method to find coefficients in the expansion (7) . We shall use a Collocation method to find the coefficints as follows:

By substituting sinc approximation expansion of unknown function $\varphi(t)$ in equation (1), we have:

$$\int_{-1}^{1} \frac{k(x,t)}{t-x} \left(\sum_{k=-N}^{N} \varphi(\mu^{-1}(kh) s(k,h)^{\circ}\mu(t) \right) dt = f(x) \qquad (9)$$

Now ,the residual function can be as follows :

$$R_N(x) = \int_{-1}^{1} \frac{k(x,t)}{t-x} \left(\sum_{k=-N}^{N} \varphi(\mu^{-1}(kh) s(k,h)^{\circ}\mu(t) \right) dt - f(x) \quad (10)$$

So, to find $\varphi(\mu^{-1}(kh))$ (which is sinc approximation expansion), we apply a collocation method with some collocation points in interval [-1,1] with a condition :

$$R_N(x_i) = 0\,, \qquad i = -N, -N+1, \ldots., N-1, N \qquad (11)$$

$$x_i = \mu^{-1}(ih) = \frac{\tan ih}{2} \qquad (12)$$

So ,

$$\int_{-1}^{1} \frac{k(x_i,t)}{t-x_i} \left(\sum_{k=-N}^{N} \varphi(\mu^{-1}(kh) s(k,h)^{\circ}\mu(t) \right) dt = f(x_i) \qquad (13)$$

Then the result of (13) gives as a system of linear algebraic equations $A_N X = B_N$,where :

$$A_N = \left[\int_{-1}^{1} \frac{k(x,t)}{t-x} (s(k,h)^{\circ}\mu(t) dt\right]_{k=-N}^{N} \qquad (14)$$

$$X^T = [\varphi(\mu^{-1}(kh))]_{K=-N}^{N} \qquad (15)$$

$$B_N = f(x_i) \quad , i = -N, -N+1, \ldots, N-1, N \qquad (16)$$

Now for evaluating matrix elements we have :

$$\int_{-1}^{1} \frac{k(x_i,t)}{t-x_i} s(k,h)^{\circ}\mu(t)dt \approx h\sum_{j=-N}^{N} \frac{k(x_i,t_j)s(k,h)}{(t_j-x_i)\mu^{-1}(t_j)} \qquad (17)$$

Where $t_j = \mu^{-1}(jh) = \frac{\tanh(jh)}{2}$, $j = -N+1, \dots, N-1, N, N+1$ (18)

4-numerical results :

In this section, we give two examples to illustrate the sinc collocation method for solving singular integral equation of the first kind with Cauchy kernel. The exact solution is known and used to show that the numerical solution obtained by with our method is good.

Example 1[8] : consider the known function $f(x) = \frac{\pi x}{1+x^2}$, and k(x,t)=1 ,then the exact solution of equation (1) is : $\varphi(x) = -\frac{\sqrt{1+x^2}}{(1+x^2)\sqrt{2}}$

Table 1 shows the absolute error between exact and numerical solution for different values of N.

N	Absolute error
5	2.33×10^{-2}
10	1.73×10^{-3}
20	4.85×10^{-3}
50	1.68×10^{-4}
100	4.37×10^{-5}

Table 1 : Absolute error for different values of N.

Example 2[2] : consider the known function

$f(x) = x^4 + 5x^3 + 2x^2 + x - \frac{11}{8}$ and k(x,t)=1 then exact solution of equation (1) in this case:

$$\varphi(x) = \frac{-1}{\pi}\sqrt{1-x^2}[x^3 + 5x^2 + \frac{5}{2}x+\frac{7}{2}]$$

Table 2: shows the absolute error between exact and numerical solution for different values of N.

N	Absolute error
5	1.38×10^{-3}
10	5.63×10^{-3}
20	2.43×10^{-4}
50	3.72×10^{-4}
100	1.86×10^{-5}

Table 2 : Absolute error for different values of N.

Conclusion:

in this paper , sinc-colloction method is used to approximate solution for singular integral equation of first kind with Cauchy kernel ,numerical result (table 1and 2) show that the error of approximate solution are very small. This show that sinc-colloction method is accurate and efficient method.

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