EXPONENETIAL AND LOGARITHMIC CRANK-NICOLSON METHODS FOR SOLVING COUPLED BURGER'S EQUATIONS

M. A. H. ALJABOORI

Collage of Education, Misan University, Iraq

Abstract

In this paper we apply a new methods called exponential and logarithmic Crank-Nicolson methods for solving a coupled of Burgers equations. These methods are shown that the stability and accuracy are better than of the exponential and logarithmic finite difference methods for long time. Error norms L_2 and L_∞ show that the present methods are a successful numerical schemes for solving a coupled Burgers' equations.

Key words: Exponential finite difference, Logarithmic finite difference, Crank-Nicolson method, Coupled Burger's equation.

1. Introduction

Nonlinear evolution equations have been the subject of intense study in various branches of sciences, mathematics, physics, biology, chemistry, engineering, etc. Moreover these equations has many applications in real life such as fluid mechanics, plasma physics, optical fibers, energy distribution, etc.

The understanding and the design of energy efficiency in buildings, (the adjustment and optimization of ventilation, heating and air conditioning), has a considerable attention in the scientific community. However, the coupling of a temperature field and a velocity field is a major problem of understanding because the complex dynamics of thermal flow in buildings. This coupling naturally involves Burger's equation and heat equation. Burger's equation has a wide variety of application in physics and engineering and is defined as,

$$u_t(t,x) + u(t,x)u_x(t,x) = \mu\, u_{xx}(t,x),$$

where $u(t,x)$ is a function in time and space and represent a velocity field and μ is the viscosity coefficient. This equation is a model that captures the interaction of convection and diffusion, so it's used to study the fluid flow. As well as this equation can be coupled with another convection diffusion equation (heat equation) to study the interaction between the temperature field and the velocity field. This couple of equations describes the incompressible fluid flow coupled to the thermal dynamics, which used to model the thermal fluid dynamics of air in building. Our system is defined by the coupled of partial differential equations, (coupled Buergers' equation),

$$u_t(t,x) + u(t,x)u_x(t,x) = \mu\, u_{xx}(t,x) - \kappa\, T(t,x) + f_1(t,x) \quad \dots(1)$$

$$T_t(t,x) + u(t,x)T_x(t,x) = \rho T_{xx}(t,x) + f_2(t,x) \quad \dots(2)$$

The function T(t, x) can be viewed as a temperature field where ρ the thermal conductivity is and κ is the coefficient of the thermal expansion, f_1 and f_2 are the forces on the system. Herein, the temperature drives the velocity field and the velocity field provides the convective term.

Bhattacharya [6], was first presented exponential finite difference method for solving 1-dimensional heat equation. This method is used to solve many partial differential equation. It predicts results with a good accuracy comparable with that obtainable by some other numerical methods. In section 2 we discretize our system by Crank-Nicolson method. Exponential and logarithmic Crank-Nicolson methods are used to solve our system in sections 3 and 4. A test problem and discuss the results are given in section 5.

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2. Crank-Nicolson method.

In this section Crank-Nicolson method used to discretize the couple. The first equation, (1), can be discretize as

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{\mu}{2}\frac{(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n)}{\Delta x^2} - U_i^n \frac{(U_{i+1}^{n+1} - U_{i-1}^{n+1} + U_{i+1}^n - U_{i-1}^n)}{4\Delta x} - \kappa T_i^n + f_{1.i}^n$$

which leads,

$$\left(\frac{-U_i^n}{4\Delta x}-\frac{\mu\Delta t}{2\Delta x^2}\right)U_{i-1}^{n+1}+\left(1+\frac{\mu\Delta t}{\Delta x^2}\right)U_i^{n+1}+\left(\frac{U_i^n}{4\Delta x}-\frac{\mu h}{2\Delta x^2}\right)U_{i+1}^{n+1}=$$
$$U_i^n+\frac{\mu}{2}\frac{(U_{i+1}^n-2U_i^n+U_{i-1}^n)}{\Delta x^2}-U_i^n\frac{(U_{i+1}^n-U_{i-1}^n)}{4\Delta x}-\kappa\Delta t T_i^n+\Delta t f_{1.i}^n\,. \quad \ldots\ldots(3)$$

For the equation (2), Crank-Nicolson method implies,

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=\frac{\rho}{2}\frac{(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1}+T_{i+1}^n-2T_i^n+T_{i-1}^n)}{\Delta x^2}-U_i^{n+1}\frac{(T_{i+1}^{n+1}-T_{i-1}^{n+1}+T_{i+1}^n-T_{i-1}^n)}{4\Delta x}+f_{2.i}^n$$

Hence,

$$\left(\frac{-U_i^{n+1}}{4\Delta x}-\frac{\mu\Delta t}{2\Delta x^2}\right)T_{i-1}^{n+1}+\left(1+\frac{\mu\Delta t}{\Delta x^2}\right)T_i^{n+1}+\left(\frac{U_i^{n+1}}{4\Delta x}-\frac{\mu\Delta t}{2\Delta x^2}\right)T_{i+1}^{n+1}$$
$$=T_i^n+\frac{\rho}{2}\frac{(T_{i+1}^n-2T_i^n+T_{i-1}^n)}{\Delta x^2}-U_i^{n+1}\frac{(T_{i+1}^n-T_{i-1}^n)}{4\Delta x}+\Delta t f_{2.i}^n$$

3. Exponential Crank-Nicolson method

In this section we discretize our system by the exponential finite difference method (Crank-Nicolson type). Firstly, multiplying equation (1) by $\frac{\partial H}{\partial u}$, where $H(u)$ is continuous function, hence

$$\frac{\partial H}{\partial t}=H'(u)[\mu\, u_{xx}(t,x)-u(t,x)u_x(t,x)-\kappa\, T(t,x)+f_1(t,x)]$$

Now discretize $\frac{\partial H}{\partial t}$ by a forward finite difference and the other terms of the last equation by the Crank-Nicolson finite difference, we have

$$H\left(u_i^{n+1}\right)=H(u_i^n)+dt\,H'(u_i^n)\left[\mu\frac{\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)+\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)}{2\Delta x^2}\right]-u_i^n\left(\frac{\left(u_{i+1}^{n+1}-u_{i-1}^{n+1}\right)+\left(u_{i+1}^n-u_{i-1}^n\right)}{4\Delta x}\right)-\kappa\,T_i^n+f_{1,i}^{n+1}$$

If take $H(u)=\ln u$, hence

$$u_i^{n+1} = u_i^n \exp\{\frac{\Delta t}{u_i^n}[(\mu \frac{(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})+(u_{i+1}^n-2u_i^n+u_{i-1}^n)}{2\Delta x^2}) - u_i^n\left(\frac{(u_{i+1}^{n+1}-u_{i-1}^{n+1})+(u_{i+1}^n-u_{i-1}^n)}{4\Delta x}\right) - \kappa T_i^n + f_{1,i}^{n+1}]\}. \quad \ldots\ldots..(5)$$

For the equation (2), suppose that $G(T)$ is continuous function. Multiplying equation (2) by $\frac{\partial G}{\partial T}$, we get

$$\frac{\partial G}{\partial t} = G'(T)[\rho\, T_{xx}(t,x) - u(t,x)T_x(t,x) + f_2(t,x)]$$

Now discretize $\frac{\partial G}{\partial t}$ by a forward finite difference and the other term by the Crank-Nicolson finite difference, we have

$$G(T_i^{n+1}) = G(T_i^n) + \Delta t\, G'(T_i^n)[\,\rho\, \frac{\frac{1}{2}(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}) + \frac{1}{2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{\Delta x^2}] - u_i^n\left(\frac{\frac{1}{2}(T_{i+1}^{n+1} - T_{i-1}^{n+1}) + \frac{1}{2}(T_{i+1}^n - T_{i-1}^n)}{2\Delta x}\right) + f_{2,i}^{n+1}$$

If take $G(T) = \ln T$, then

$$T_i^{n+1} = T_i^n \exp\{\frac{\Delta t}{T_i^n}[(\rho\, \frac{\frac{1}{2}(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1})+\frac{1}{2}(T_{i+1}^n-2T_i^n+T_{i-1}^n)}{\Delta x^2}) - T_i^n\left(\frac{\frac{1}{2}(T_{i+1}^{n+1}-T_{i-1}^{n+1})+\frac{1}{2}(T_{i+1}^n-T_{i-1}^n)}{2\Delta x}\right) + f_{2,i}^{n+1}]\} \ldots\ldots.(6)$$

4. Logarithmic finite difference method (Crank-Nicolson type).

Multiplying equation (1) by $\frac{\partial H}{\partial u}$, where $H(u)$ is continuous function.

$$\frac{\partial H}{\partial t} = H'(u)[\mu\, u_{xx}(t,x) - u(t,x)u_x(t,x) - \kappa\, T(t,x) + f_1(t,x)]$$

Now discretize $\frac{\partial H}{\partial t}$ by a forward finite difference and the other term by the Crank-Nicolson finite difference, we have

$$H\left(u_i^{n+1}\right) = H(u_i^n) + \Delta t\, H'(u_i^n)[\,\mu\, \frac{\frac{1}{2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{\Delta x^2}] - u_i^n \left(\frac{\frac{1}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) + \frac{1}{2}(u_{i+1}^n - u_{i-1}^n)}{2\Delta x}\right) - \kappa\, T_i^n + f_{1,i}^{n+1}$$

Take $H(u) = e^u$, then

$$u_i^{n+1} = u_i^n + \ln\{\Delta t[(\,\mu\, \frac{\frac{1}{2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{\Delta x^2}) - u_i^n \left(\frac{\frac{1}{2}(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + \frac{1}{2}(u_{i+1}^n - u_{i-1}^n)}{2\Delta x}\right) - \kappa\, T_i^n + f_{1,i}^{n+1}]\}$$

……(7)

For the second equation, multiplying equation (2) by $\frac{\partial G}{\partial T}$, where $G(T)$ is continuous function, we get

$$\frac{\partial G}{\partial t} = G'(T)[\rho\, T_{xx}(t,x) - u(t,x)T_x(t,x) + f_2(t,x)]$$

By the same way, we obtain

$$T_i^{n+1} =$$
$$T_i^n + \ln\{\Delta t[(\upsilon \frac{\frac{1}{2}(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1})+\frac{1}{2}(T_{i+1}^{n}-2T_i^{n}+T_{i-1}^{n})}{\Delta x^2}) - u_i^n\left(\frac{\frac{1}{2}(T_{i+1}^{n+1}-T_{i-1}^{n+1})+\frac{1}{2}(T_{i+1}^{n}-T_{i-1}^{n})}{2\Delta x}\right) + f_{2,i}^{n+1}]$$
$$\} \quad ....(8)$$

The nonlinear systems (5) - (8) can be solved by Newton's method, so we consider every system of 5-8 as

$$H(U) = 0,$$

where $H = [h_1, h_2, \dots, h_{N-1}]^T$and $U = [U_1^{n+1}, U_2^{n+1}, \dots, U_{N-1}^{n+1}]^T$, (here T is the transpose matrix). Firstly, we set U^0 as initial approximation. Then solve

$$J(U^m)\delta^m = -H(V^m),$$

and set $U^{m+1} = U^m + \delta^m$, for $m = 1,2,\dots$ until convergent, where $J(U^m)$ is the Jacobian matrix which can be computed analytically.

The consistency and accuracy of the schemes are measured in terms of error norms L_2 and L_∞ defined as:

$$L_2 = \| u - u_N \|_2 = \left(\sum_{i=1}^{N} |(u_i - (u_N)_i|^2\right)^{1/2}$$
$$L_\infty = \| u - u_N \|_\infty = \max_i |u_i - (u_N)_i|$$

where u and u_N are exact and computed solutions respectively.

5. Numerical Experiment

For a particular case, we consider the following coupled of equation

$$u_t(t,x) + u(t,x)u_x(t,x) = \mu\, u_{xx}(t,x) - \kappa\, T(t,x) + f_1(t,x)$$

$$T_t(t,x) + u(t,x)T_x(t,x) = \rho\, T_{xx}(t,x) + f_1(t,x)$$

subject to the initial condition

$$u(0,x) = sinx,$$

$$T(0,x) = \frac{1}{2} sin2x$$

the exact solution is given by

$$u(0,x) = e^{-t} sinx,$$

$$T(0,x) = \frac{1}{2} e^{-2t} sin2x$$

where $\mu = \rho = \kappa = 1$ and

$$f_1 = e^{-2t} sin2x , \qquad f_2 = e^{-3t} sinx\, cos2x + e^{-2t} sin2x$$

In this example we compare between the exact solution (u, T) with exponential Finite difference(u_e, T_e), logarithmic finite difference (u_l, T_l), exponential Crank-Nicolson (u_{ec}, T_{ec}) and logarithmic Crank-Nicolson method (u_{lc}, T_{lc}), at different times.

Table 1. comparison between the exact solution u and the numerical solutions u_e, u_l, u_{ec} and u_{lc} at $t = 0.1$

x	u	u_e	u_l	u_{ec}	u_{lc}
0	0	0.0000011457	0.0000008691	0.0000000378	0.0000000634
0.2618	0.1298751901	0.1298799693	0.1298715796	0.1298756685	0.1298755471
0.5236	0.2575281797	0.2575210394	0.2575295875	0.2575288954	0.2575284318
0.7854	0.3807747907	0.3807798495	0.3807275901	0.3807742875	0.3807742194
1.0472	0.4975062395	0.4975068305	0.4975065465	0.4975069256	0.4975062148
1.3090	0.6057252187	0.6057237858	0.6057558768	0.6057259685	0.6057259475
1.5708	0.7035800714	0.7035808379	0.7035881980	0.7035801869	0.7035801885
1.8326	0.7893964739	0.7893999650	0.7893926187	0.7893969472	0.7893969246
2.0944	0.8617060840	0.8617074048	0.8617021896	0.8617063918	0.8617069452
2.3562	0.9192716641	0.9192720496	0.9192715548	0.9192765287	0.9192713108
2.6180	0.9611082511	0.9611065681	0.9611007317	0.9611088286	0.9611059195
2.8798	0.9865000094	0.9865051956	0.9865003439	0.9865002146	0.9865001985
3.1416	0.9950124791	0.9950120568	0.9950193749	0.9950122166	0.9950125186
3.4034	0.9865000094	0.9865001085	0.9865024516	0.9865009284	0.9865003299
3.6652	0.9611082511	0.9611024597	0.9611009605	0.9611088346	0.9611068257
3.9270	0.9192716641	0.9192798036	0.9192765479	0.9192713098	0.9192711615
4.1888	0.8617060840	0.8617084759	0.8617018653	0.8617069247	0.8617061854
4.4506	0.7893964739	0.7893969588	0.7893970361	0.7893961829	0.7893963846
4.7124	0.7035800714	0.7035429830	0.7035807610	0.7035802180	0.7035819579
4.9742	0.6057252187	0.6057969902	0.6057278395	0.6057258526	0.6057253581
5.2360	0.4975062395	0.4975004681	0.4975053108	0.4975061568	0.4975061680
5.4978	0.3807747907	0.3807798106	0.3807707643	0.3807749824	0.3807765489
5.7596	0.2575281797	0.2575197409	0.2575087768	0.2575281685	0.2575287154
6.0214	0.1298751901	0.1298740986	0.1298709715	0.1298759345	0.1298758496

6.2832	0	0.0000006139	0.0000084620	0.0000000247	0.0000004853
L_2		8.313288e-05	6.247992e-05	5.439926e-06	4.172784e-06
L_∞		7.17715E-05	4.72006E-05	4.8646E-06	2.3316E-06

Table 2. comparison between the exact solution u and the numerical solutions u_e, u_l, u_{ec} and u_{lc} at $t = 0.5$

x	u	u_e	u_l	u_{ec}	u_{lc}
0	0	0.000008419	0.000093126	0.0000008354	0.0000083618
0.2618	0.127303489	0.1273145790	0.1273482561	0.1273031545	0.1273095854
0.5236	0.252428780	0.2524816186	0.2524868115	0.2524257898	0.2524252548
0.7854	0.373234944	0.3732331847	0.3732336942	0.3732385429	0.3732363584
1.0472	0.487654956	0.4876584515	0.4876953697	0.4876543182	0.4876528455
1.3090	0.593731055	0.5937384540	0.5937165864	0.5937361520	0.5937335840
1.5708	0.689648252	0.6896025756	0.6895225475	0.6896452147	0.6896433921
1.8326	0.773765376	0.7737484739	0.7737973198	0.7737653658	0.7737698610
2.0944	0.844643160	0.8443124587	0.8449741856	0.8446495576	0.8446465842
2.3562	0.901068865	0.9012548605	0.9019483142	0.9010683581	0.9010669541
2.6180	0.942077032	0.9427495258	0.9423932941	0.9420796584	0.9420776975
2.8798	0.966966000	0.9669251763	0.9669118594	0.9669661547	0.9669662547
3.1416	0.975309912	0.9757860396	0.9756832978	0.9753099685	0.9753084576
3.4034	0.966966000	0.9668689252	0.9669182876	0.9669663989	0.9669636954
3.6652	0.942077032	0.9427675846	0.9427084650	0.9420739745	0.9420764780
3.9270	0.901068865	0.9010275864	0.9010163651	0.9010696584	0.9010692475
4.1888	0.844643160	0.8446864857	0.8446958486	0.8446436685	0.8446491041
4.4506	0.773765376	0.7736087346	0.7737036584	0.7737284675	0.7737696711
4.7124	0.689648252	0.6896357061	0.6896185428	0.6896469847	0.6896497845
4.9742	0.593731055	0.5938328643	0.5930317930	0.5937336975	0.5937305566
5.2360	0.487654956	0.4871359687	0.4876128480	0.4876569531	0.4876515796
5.4978	0.373234944	0.3733854287	0.3732284652	0.3732364128	0.3732361745
5.7596	0.252428780	0.2523057584	0.2528986284	0.2524221764	0.2524236175
6.0214	0.127303489	0.1273653801	0.1273031864	0.1273039687	0.1273096574
6.2832	0	0.0000057384	0.0000082650	0.0000062861	0.000007258
L_2		0.001290690	0.001510243	3.921290e-05	1.8271313e-05
L_∞		6.905520e-04	8.794487e-04	3.6909E-05	8.3618000e-06

Table 3. comparison between the exact solution u and the numerical solutions u_e, u_l, u_{ec} and u_{lc} at $t = 5$

	u	u_e	u_l	u_{ec}	u_{lc}
0	0	0.000834752	0.0000258471	0.000015948	0.000045810
0.261	0.1016539007	0.1048679035	0.1027135184	0.101653013	0.1016535694
0.523	0.2015684749	0.204765855	0.2036175249	0.201521685	0.2015945561
0.785	0.2980341567	0.200821859	0.2911310254	0.298015389	0.2980367458
1.047	0.3894003915	0.353568948	0.3819648721	0.389491843	0.3894153601
1.309	0.4741038776	0.474254826	0.4737658743	0.474154871	0.4741256539
1.570	0.5506953149	0.550125657	0.5532478516	0.550695489	0.5506475821
1.832	0.6178642026	0.657482785	0.6116584932	0.617836578	0.6178847566
2.094	0.6744612626	0.675868104	0.6794545486	0.674468752	0.6744865341

2.356	0.7195181033	0.711848584	0.7172145879	0.7195471521	0.7195161658
2.618	0.7522637899	0.758547849	0.7505487963	0.7522218625	0.7522656856
2.879	0.7721380344	0.730689475	0.7782569874	0.7721316194	0.7721258746
3.141	0.7788007830	0.778039584	0.7743677415	0.7788375895	0.7788065848
3.403	0.7721380344	0.778285485	0.7716984210	0.7721019852	0.7721975854
3.665	0.7522637899	0.757845254	0.7722345679	0.7522630899	0.7522915748
3.927	0.7195181033	0.712054206	0.7096127845	0.7195541657	0.7195214782
4.188	0.6744612626	0.678917846	0.6754871592	0.6744645879	0.6744565895
4.450	0.6178642026	0.610371485	0.6171854265	0.6178639875	0.6178681582
4.712	0.5506953149	0.551617519	0.5512133584	0.5506275698	0.5506919525
4.974	0.4741038776	0.474558743	0.4786054582	0.4741054867	0.4741095816
5.236	0.3894003915	0.383392848	0.3890224869	0.3894658492	0.3894196565
5.497	0.2980341567	0.291284658	0.2984256982	0.2980325479	0.2980495876
5.759	0.2015684749	0.205392854	0.2012365897	0.2015864789	0.2015842658
6.021	0.1016539007	0.108011458	0.1059587492	0.1016352698	0.1016552689
6.283	0	0.000017465	0.0002372843	0.0000048579	0.0000002574
L_2		0.120343710	0.027960701	1.753269e-04	1.1083223e-04
L_∞		0.097212297	0.019970778	5.09937E-05	5.9551E-05

Table 4. comparison between the exact solution and the numerical solutions T_e, T_l, T_{ec} and T_{lc} at $t = 0.1$

x	T	T_e	T_l	T_{ec}	T_{lc}
0	0	0.0000038501	0.0000008150	0.0000009154	0.0000007548
0.261	0.1281218762	0.1281237930	0.1281285667	0.1281217532	0.1281218316
0.523	0.2475124584	0.2475149875	0.2475157842	0.2475122789	0.2475124817
0.785	0.3500354755	0.3500358769	0.3500351795	0.3500357578	0.3500355240
1.047	0.4287041535	0.4287093757	0.4287096536	0.4287043960	0.4287042135
1.309	0.4781573518	0.4781586970	0.4781528653	0.4781573467	0.4781570897
1.570	0.4950249168	0.4950241686	0.4950249875	0.4950249245	0.4950248509
1.832	0.4781573518	0.4781575154	0.4781576734	0.4781575678	0.4781574689
2.094	0.4287041535	0.4287048542	0.4287037694	0.4287042867	0.4287048954
2.356	0.3500354755	0.3500385566	0.3500364586	0.3500354697	0.3500354315
2.618	0.2475124584	0.2475128086	0.2475198667	0.2475128965	0.2475124718
2.879	0.1281218762	0.1281217552	0.1281216459	0.1281215750	0.1281218798
3.141	0	0.0000395749	0.0000001387	0.0000043904	0.0000092130
3.403	-0.1281218762	-0.1281267009	-0.1281214367	-0.1281218354	-0.1281218528
3.665	-0.2475124584	-0.2475168750	-0.2475187987	-0.2475124793	-0.2475122584
3.927	-0.3500354755	-0.3500354439	-0.3500359865	-0.3500353487	-0.3500354625
4.188	-0.4287041535	-0.4287041768	-0.4287687422	-0.4287044513	-0.4287041254
4.450	-0.4781573518	-0.4781571657	-0.4781518643	-0.4781573638	-0.4781573820
4.712	-0.4950249168	-0.4950247632	-0.4950249875	-0.4950242874	-0.4950249942
4.974	-0.4781573518	-0.4781529641	-0.4781573867	-0.4781573584	-0.4781573808
5.236	-0.4287041535	-0.4287342345	-0.4287055633	-0.4287041478	-0.4287045284
5.497	-0.3500354755	-0.3500344698	-0.3500350274	-0.3500359856	-0.3500358641
5.759	-0.1281218762	-0.1281198589	-0.1281245345	-0.1281212787	-0.1281218824
6.021	-0.1281218762	-0.1281210754	-0.1281223457	-0.1281219854	-0.1281218575
6.283	0	0.0000062879	0.00000193670	0.0000007391	0.0000009584
L_2		3.221068e-05	6.644593e-05	1.5821940e-06	1.2484795e-06

L_∞		3.008099e-05	6.458869e-05	9.1540000e-07	7.5480000e-07

Table 5. comparison between the exact solution and the numerical solutions T_e, T_l, T_{ec} and T_{el} at $t = 0.5$

x	T	T_e	T_l	T_{ec}	T_{lc}
0	0	0.0000925610	0.0000044831	0.0000005821	0.0000038549
0.2618	0.1230981456	0.1230856838	0.1236941471	0.1230926387	0.1230982764
0.5236	0.2378073561	0.2378284068	0.2378503851	0.2378076373	0.2378072938
0.7854	0.3363103882	0.3363217058	0.3369525847	0.3363196380	0.3363109403
1.0472	0.4118944232	0.4118038679	0.4118065584	0.4118946849	0.4118947384
1.3090	0.4594085339	0.4594018566	0.4594847853	0.4594037495	0.4594063484
1.5708	0.4756147122	0.4756174058	0.4756285445	0.4756147389	0.4756146093
1.8326	0.4594085339	0.4594029374	0.4594375466	0.4594015908	0.4594087369
2.0944	0.4118944232	0.4118934947	0.4119541840	0.4118943804	0.4118942837
2.3562	0.3363103882	0.3363245975	0.3363475568	0.3363107649	0.3363118474
2.6180	0.2378073561	0.2378358657	0.2378296755	0.2378059084	0.2378074859
2.8798	0.1230981456	0.1230958076	0.1230179652	0.1230982587	0.1230986384
3.1416	0	0.0000046801	0.0000035467	0.0000009741	0.00000846254
3.4034	-0.123098145	-0.1230369564	-0.1231476586	-0.1230912706	-0.1230983608
3.6652	-0.237807356	-0.2378052861	-0.2372179575	-0.2378014750	-0.2378073449
3.9270	-0.336310388	-0.3363265582	-0.3363103882	-0.3363109576	-0.3363118474
4.1888	-0.411894423	-0.4118047912	-0.4118942356	-0.4118934924	-0.4118943657
4.4506	-0.459408533	-0.4594686085	-0.4594565831	-0.4594084863	-0.4594125738
4.7124	-0.475614712	-0.4753025755	-0.4756296760	-0.4756193758	-0.4756144590
4.9742	-0.459408533	-0.4594082879	-0.4594060834	-0.4594053758	-0.4594036480
5.2360	-0.411894423	-0.4190925842	-0.4188903875	-0.4118937586	-0.4118797438
5.4978	-0.336310388	-0.3369649865	-0.3365708430	-0.3363187384	-0.3363100389
5.7596	-0.237807356	-0.2378244860	-0.2378548159	-0.2378019405	-0.2378072847
6.0214	-0.123098145	-0.1232765679	-0.1234189341	-0.1230981386	-0.1230957495
6.2832	0	0.0000093640	0.0000068654	0.0000008643	0.0000066521
L_2		0.0072391946	0.007089774	2.0141552e-05	2.0054330e-05
L_∞		0.007198161	0.006995964	9.2497999e-06	1.4679400e-05

Table 6. comparison between the exact solution and the numerical solutions T_e, T_l, T_{ec} and T_{lc} at $t = 5$

x	T	T_e	T_l	T_{ec}	T_{lc}
0	0	0.0000934781	0.0000084698	0.0000068261	0.0000008470
0.261	0.078490843	0.0119759503	0.0710483984	0.0784830558	0.0784918503
0.523	0.151632664	0.1288589484	0.1510474886	0.1516694756	0.1516383467
0.785	0.214440971	0.2574858485	0.2199847398	0.2144023654	0.2144092870
1.047	0.262635479	0.2058594849	0.2863499720	0.2626303959	0.2626915669
1.309	0.292931814	0.2748594924	0.2985008305	0.2929310473	0.2929254786
1.570	0.303265329	0.3184396811	0.3574968074	0.3032304876	0.3032648264
1.832	0.292931814	0.2506487750	0.2973957308	0.2929330725	0.2929325481
2.094	0.262635479	0.2073408857	0.2663947795	0.2626294672	0.2626396478
2.356	0.214440971	0.2109137871	0.2146495771	0.2144409247	0.2144411857
2.618	0.151632664	0.1948473994	0.1597745300	0.1516819633	0.1516258485
2.879	0.078490843	0.0716008453	0.0649584374	0.0784603857	0.0784952480

3.141	0	0.000063848	0.000639460	0.0000183692	0.000007246
3.403	-0.078490843	-0.037565998	-0.004987585	-0.0781915646	-0.078425486
3.665	-0.151632664	-0.185075990	-0.172749639	-0.1516305438	-0.151615867
3.927	-0.214440971	-0.298605867	-0.268408303	-0.2144486307	-0.214442251
4.188	-0.262635479	-0.207498579	-0.262485708	-0.2624735902	-0.262631548
4.450	-0.292931814	-0.237357730	-0.229485079	-0.2929295740	-0.292932548
4.712	-0.303265329	-0.307385975	-0.365158744	-0.3032698454	-0.303261547
4.974	-0.292931814	-0.211875609	-0.292189495	-0.2929323905	-0.292932548
5.236	-0.262635479	-0.266739479	-0.228567499	-0.2626352146	-0.262647585
5.497	-0.214440971	-0.215498346	-0.263857974	-0.2144451624	-0.214476384
5.759	-0.151632664	-0.137419504	-0.128945754	-0.1516856989	-0.151635879
6.021	-0.078490843	-0.093474309	-0.069843567	-0.0784955486	-0.078496530
6.283	0	0.000063246	0.000064857	0.0000061806	0.00000197
L_2		0.200962834	0.1572099397	3.55923432e-04	1.0193112e-04
L_∞		0.084164896	0.0735032571	2.99278404e-04	6.5357000e-05

5.1 Discussion

Approximate solution of the coupled Burger's equation have been preformed by using the exponential and logarithmic Crank-Nicolson method in the domain $[0, 2\pi]$, , where the initials are functions of the variable x, and the external forces f_1 and f_2 are given. We compute the error norms L_2 and L_∞ at times $t = 0.1, t = 0.5$ and $t = 5$ with $\Delta t = 0.001$ as shown in tables 1-6. The results show that, the stability of these methods are better than the stability of exponential and logarithmic finite difference. Furthermore, error norms L_2 and L_∞ show that, the schemes of these methods are more consistence and accurate than the others.

6. Conclusion

This work is employing to show the powerful of the exponential and logarithmic Crank-Nicolson methods for solving a nonlinear partial differential equation, particularly, coupled of Burger's equation. From the tables of solution at different time, we see that these methods are stable for long time more than the finite difference, exponential finite difference logarithmic finite difference. Moreover, this method is quite efficient and suitable for finding the approximate solution of this couple of PDE.

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المستخلص

في هذا البحث نطبق طرق جديدة تسمى طريقة كرانك نيكولسون الاسية وطريقة كرانك نيكولسون اللوغارتمية لحل زوج من معادلتي برجر. وقد اوضحنا ان استقرارية ودقة هذه الطرق هي افضل من طريقتي الفروقات المنتهية الاسية واللوغارتمية للاوقات الكبيرة. معياري الخطأ L_2 و L_∞ يوضحان بان هذه الطرق ناجحة لحل هذا النوع من المعادلات التفاضلية الجزئية.